Almost Necessary

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(based on joint work with Yanjing Wang and Hans van Ditmarsch)

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Reference

- J. Fan, Y. Wang and H. van Ditmarsch. Almost Necessary. Advances in Modal Logic, Groningen, The Netherlands, Volume 10, pages 178-196, 2014.
- J. Fan, Y. Wang and H. van Ditmarsch. Contingency and Knowing Whether. To appear in *The Review of Symbolic Logic*.

Contingency: an example

- Will there be sea battles tomorrow? No!
- Will there be no sea battles tomorrow? No!
- Why is it the case?
- The proposition "there will be sea battles tomorrow" (P) is **contingent**, i.e., it is possible that P and it is possible that not P.

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- Doxastic Logic: neither believe nor believe-not, i.e. agnosticism

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- ...

Relating necessity and (non-)contingency

- (Non-)contingency can be defined with necessity [Montgomery and Routley, 1966]:

$$\Delta\varphi =_{df} \Box\varphi \vee \Box\neg\varphi.$$

$$\nabla\varphi =_{df} \neg\Delta\varphi.$$

Δ: non-contingency
∇: contingency
$\Box$: necessity

Q1: the definability of $\Box$

- Is $\Box$ definable in terms of Δ?
  1. $\Box\varphi =_{df} \Delta\varphi \wedge \varphi$ [Montgomery and Routley, 1966], only in the systems containing $\Box\varphi \to \varphi$ [Segerberg, 1982, page 128]
  2. $\Box$ can only be defined in terms of Δ in the Verum system, or the systems containing $\Box\varphi \to \Diamond\varphi$ [Cresswell, 1988]
  3. With Contingency Postulate, $\Box\varphi =_{df} \forall p(\Delta(p \wedge \varphi) \to \Delta p)$ [Pizzi, 1999]
  4. With the axiom $\nabla\tau$ (τ is a propositional constant), $\Box\varphi =_{df} \Delta\varphi \wedge \Delta(\tau \to \varphi)$ [Pizzi, 2006, Pizzi, 2007]
  5. $\boxtimes\varphi =_{df} \bigwedge_{\psi \in \mathbf{NCL}} \Delta(\psi \to \varphi)$, $\boxtimes$ behaves like, but differs from $\Box$ [Zolin, 2001]

None of them is satisfactory!

Q2: fragment

- $\Delta\varphi =_{df} \Box\varphi \vee \Box\neg\varphi$
- **NCL** $\subseteq$ **ML**
- What is the exact fragment?

Q3: axiomatizing **NCL** over symmetric frames

Usual frame classes	Known results
$\mathcal{K}$	[Humberstone, 1995, Kuhn, 1995, Zolin, 1999]
$\mathcal{D}$	[Humberstone, 1995, Zolin, 1999]
$\mathcal{T}$	[Montgomery and Routley, 1966]
4	[Kuhn, 1995, Zolin, 1999]
5	[Zolin, 1999]
$\mathcal{B}$	?

Main results of the paper

- Is $\Box$ definable in terms of Δ in the general case?
  —— almost-definability schema
- How to characterize non-contingency logic within modal logic?
  —— Δ-bisimulation and characterization fragement
- axiomatizing **NCL** on $\mathcal{B}$
  —— axiomatization $\mathbb{NCLB}$ and completeness
- Multimodal Non-contingency Logic: axiomatization and completeness

Non-contingency logic: Language

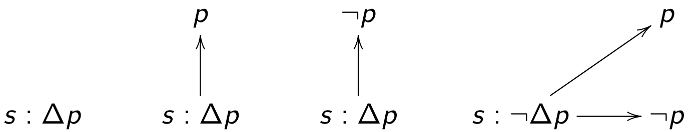
NCL $\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta\varphi$

ML $\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$

- $\Delta\varphi$: it is non-contingent that φ.
- $\nabla\varphi =_{df} \neg\Delta\varphi$: it is contingent that φ.

Non-contingency logic: Semantics

$$\mathcal{M}, s \vDash \Delta\varphi \quad \Leftrightarrow \quad \text{for any } t_1, t_2 \text{ such that } sRt_1, sRt_2: (\mathcal{M}, t_1 \vDash \varphi \Leftrightarrow \mathcal{M}, t_2 \vDash \varphi)$$

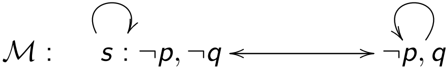


Non-contingency logic is not normal

Even on **S5**-models,

$$\not\models \Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$$

e.g.,



$\mathcal{M}, s \models \Delta(p \to q) \wedge \Delta p$, but $\mathcal{M}, s \not\models \Delta q$.

Almost-definability

- Under a condition $\nabla\psi$ for some ψ, $\Box$ is definable with Δ

Proposition

$$\vDash \nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \land \Delta(\psi \to \varphi))$$

Standard bisimulation

Definition ($\Box$-Bisimulation)

Let $\mathcal{M} = \langle S, R, V \rangle$ and $\mathcal{M}' = \langle S', R', V' \rangle$ be two models. A binary relation Z is a $\Box$-*bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$, if Z is non-empty and whenever sZs':

- (Invariance) s and s' satisfy the same propositional variables;
- ($\Box$-Zig) if sRt, then there is a t' in $\mathcal{M}'$ such that $s'R't'$ and tZt';
- ($\Box$-Zag) if $s'R't'$, then there is a t in $\mathcal{M}$ such that sRt and tZt'.

We say that $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are $\Box$-*bisimilar*, if there is a $\Box$-bisimulation linking two states s in $\mathcal{M}$ and s' in $\mathcal{M}'$, and we write $(\mathcal{M}, s) \leftrightarroweq_{\Box} (\mathcal{M}', s')$.

$\Box$-bisimulation is too strong for **NCL**

Example

$\mathcal{M}$: $s : p \longrightarrow t : p$

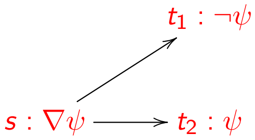
$\mathcal{M}'$: $s' : p$

$$(\mathcal{M}, s) \equiv_{\mathbf{NCL}} (\mathcal{M}', s') \text{ but } (\mathcal{M}, s) \not\simeq_{\Box} (\mathcal{M}', s')$$

Almost-definability: Revisited

$$\nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi))$$

- Under a condition $\nabla\psi$ for some ψ, $\Box$ is definable with Δ



- t_1 and t_2 are non-**NCL**-equivalent (Semantic reading)
- (t_1, t_2) are not bisimilar (Structural counterpart)

Bisimulation for **NCL**

Definition (Δ-Bisimulation)

Let $\mathcal{M} = \langle S, R, V\rangle$ be a model. A binary relation Z over S is a Δ-*bisimulation* on $\mathcal{M}$, if Z is non-empty and whenever sZs':

- (Invariance) s and s' satisfy the same propositional variables;
- (Δ-Zig) if there are two successors t_1, t_2 of s such that $(t_1, t_2) \notin Z$ and sRt, then there is a t' such that $s'Rt'$ and tZt';
- (Δ-Zag) if there are two successors t'_1, t'_2 of s' such that $(t'_1, t'_2) \notin Z$ and $s'Rt'$, then there is a t such that sRt and tZt'.

We say $(\mathcal{M}, s)$ and $(\mathcal{M}', s')$ are Δ-*bisimilar*, notation: $(\mathcal{M}, s) \leftrightarrows_\Delta (\mathcal{M}', s')$, if there is a Δ-bisimulation linking s and s' in the disjoint union of $\mathcal{M}$ and $\mathcal{M}'$.

Δ-bisimilarity: an example

Example

$$\mathcal{M}: \qquad s:p \longrightarrow t:p$$

$$\mathcal{M}': \qquad s':p$$

$$(\mathcal{M}, s) \equiv_{\mathbf{NCL}} (\mathcal{M}', s') \text{ but } (\mathcal{M}, s) \not\leftrightarroweq_{\Box} (\mathcal{M}', s')$$

$$(\mathcal{M}, s) \leftrightarroweq_{\Delta} (\mathcal{M}', s')$$

$\underline{\leftrightarrow}_\Delta$ is strictly weaker than $\underline{\leftrightarrow}_\Box$

Proposition

- $(\mathcal{M}, s) \underline{\leftrightarrow}_\Box (\mathcal{N}, t) \Longrightarrow (\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{N}, t)$
- $(\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{N}, t) \not\Longrightarrow (\mathcal{M}, s) \underline{\leftrightarrow}_\Box (\mathcal{N}, t)$

$\underline{\leftrightarrow}_\Delta$ is suitable for **NCL**

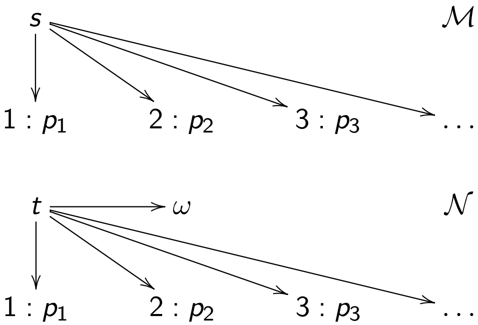
Proposition

- $(\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{M}', s') \Longrightarrow (\mathcal{M}, s) \equiv_{\mathbf{NCL}} (\mathcal{M}', s')$
- *For any* **NCL**-*saturated pointed models* $(\mathcal{M}, s)$ *and* $(\mathcal{N}, t)$, $(\mathcal{M}, s) \underline{\leftrightarrow}_\Delta (\mathcal{N}, t) \Longleftrightarrow (\mathcal{M}, s) \equiv_{\mathbf{NCL}} (\mathcal{N}, t)$

- A model $\mathcal{M}$ is said to be **NCL-saturated**, if given any $s \in \mathcal{M}$, any $\Sigma \subseteq$ **NCL**, if Σ is finitely satisfiable in $R(s)$, then Σ is satisfiable in $R(s)$.

NCL-saturation is necessary

Without the condition '**NCL**-saturation',
$(\mathcal{M}, s) \equiv_{\textbf{NCL}} (\mathcal{N}, t) \not\Longrightarrow (\mathcal{M}, s) \leftrightarrows_{\Delta} (\mathcal{N}, t)$.



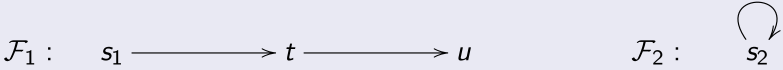
Applications of Δ-bisimulation

Proposition

1. *The property "is an endpoint" is undefinable in* **NCL**.
2. *The frame properties of seriality, reflexivity, transitivity, symmetry, and Euclidicity are not definable in* **NCL**.
3. **NCL** *is less expressive than* **ML** *on the class of symmetric (and many other) models.*

Proof.

Take 2 for instance.



□

Two characterization results

Theorem

An **ML***-formula is equivalent to an* **NCL***-formula iff it is invariant under* Δ*-bisimulation.*

Theorem

A first-order formula is equivalent to an **NCL***-formula iff it is invariant under* Δ*-bisimulation.*

Proof system for $\mathcal{K}$-frames

$\mathbb{NCL}$:

TAUT	all instances of tautologies
ΔCon	$\Delta(\chi \to \varphi) \wedge \Delta(\neg\chi \to \varphi) \to \Delta\varphi$
ΔDis	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$
ΔEqu	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
MP	From φ and $\varphi \to \psi$ infer ψ
NECΔ	From φ infer $\Delta\varphi$
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta\varphi \leftrightarrow \Delta\psi$

NB: NECΔ is indispensable in $\mathbb{NCL}$

Proposition

$\mathbb{NCL}$ *is sound with respect to the class of* $\mathcal{K}$*-frames.*

Proof methods for the completeness of $\mathbb{NCL}$: an overview

- **NCL** is not normal, and the usual frame properties are undefinable in **NCL**, which make the completeness proof non-trivial

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- Kuhn [Kuhn, 1995]: $sR^c t$ iff $\lambda(s) \subseteq t$, where $\lambda(s) = \{\varphi \mid \text{ for every } \psi \in \textbf{NCL}, \Delta(\varphi \vee \psi) \in s\}$

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- Zolin [Zolin, 1999]: sR^ct iff $\sharp(s) \subseteq t$, where $\sharp(s) = \{\varphi \mid \boxtimes\varphi \subseteq s\}$, in which $\boxtimes\varphi = \{\Delta(\psi \to \varphi) \mid \psi \in \mathbf{NCL}\}$

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- NB: Kuhn's λ and Zolin's $\sharp$ are the same function

Limitations

- Humberstone [Humberstone, 1995]:
  $\lambda(s) = \{\varphi \mid \Delta\varphi \in s \text{ and for all } \psi, \vdash \varphi \to \psi \text{ implies } \Delta\psi \in s\}$
  is responsible for the infinitary axiomatization, and the completeness proof requires König's Lemma
- Kuhn [Kuhn, 1995] and Zolin [Zolin, 1999]:
  - The necessity operator, defined by $\boxtimes\varphi =_{df} \bigwedge_{\psi \in \mathbf{NCL}} \Delta(\varphi \vee \psi)$, is not really $\Box$. E.g., $\varphi \to \boxtimes\neg\boxtimes\neg\varphi$ is not valid on the class of symmetric frames [Zolin, 2001].
  - The canonical relations in [Kuhn, 1995, Zolin, 1999] at least do not apply to the reflexive frames, a fortiori, they do not apply to the symmetric frames [Humberstone, 2002, page 118].

A uniform method

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Definition (Canonical model)

Define $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$ as follows:

- $S^c = \{s \mid s \text{ is a maximal consistent set of } \mathbb{NCL}\}$
- For all $s, t \in S^c$, $sR^c t$ iff there exists χ such that:
  - $\neg\Delta\chi \in s$, and
  - for all φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ implies $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

- Standard modal logic: $sR^c t$ iff for all φ, $\Box\varphi \in s$ implies $\varphi \in t$
- Almost-definability: $\neg\Delta\chi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\chi \to \varphi))$

Completeness for $\mathbb{NCL}$

Lemma

For all $\varphi \in$ **NCL** *and* $s \in S^c$, $\mathcal{M}^c, s \vDash \varphi$ *iff* $\varphi \in s$.

Theorem

$\mathbb{NCL}$ *is complete with respect to the class of* $\mathcal{K}$*-frames.*

Axiomatization: extensions

Notation	Axiom Schemas	Systems
$\Delta\mathtt{T}$	$\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \varphi \to \Delta\psi$	$\mathbb{NCLT} = \mathbb{NCL} + \Delta\mathtt{T}$
$\Delta 4$	$\Delta\varphi \to \Delta(\Delta\varphi \vee \psi)$	$\mathbb{NCL}4 = \mathbb{NCL} + \Delta 4$
$\Delta 5$	$\neg\Delta\varphi \to \Delta(\neg\Delta\varphi \vee \psi)$	$\mathbb{NCL}5 = \mathbb{NCL} + \Delta 5$
$\Delta\mathtt{B}$	$\varphi \to \Delta((\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi)$	$\mathbb{NCLB} = \mathbb{NCL} + \Delta\mathtt{B}$
$\mathtt{w}\Delta 4$	$\Delta\varphi \to \Delta\Delta\varphi$	$\mathbb{NCLS}4 = \mathbb{NCLT} + \mathtt{w}\Delta 4$
$\mathtt{w}\Delta 5$	$\neg\Delta\varphi \to \Delta\neg\Delta\varphi$	$\mathbb{NCLS}5 = \mathbb{NCLT} + \mathtt{w}\Delta 5$

- $\mathtt{NEC}\Delta$ is admissible in $\mathbb{NCLB}$, different from other systems
- Completeness results w.r.t. corresponding classes of frames
- apply to multimodal cases, except for that of $\mathbb{NCLB}$

Proof system for the symmetric frames

$\mathbb{NCLB}$: (NB: no need of the rule ($\texttt{NEC}\Delta$): $\frac{\varphi}{\Delta\varphi}$)

TAUT	all instances of tautologies
ΔCon	$\Delta(\chi \to \varphi) \wedge \Delta(\neg\chi \to \varphi) \to \Delta\varphi$
ΔDis	$\Delta\varphi \to \Delta(\varphi \to \psi) \vee \Delta(\neg\varphi \to \chi)$
ΔEqu	$\Delta\varphi \leftrightarrow \Delta\neg\varphi$
ΔB	$\varphi \to \Delta((\Delta\varphi \wedge \Delta(\varphi \to \psi) \wedge \neg\Delta\psi) \to \chi)$
MP	From φ and $\varphi \to \psi$ infer ψ
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta\varphi \leftrightarrow \Delta\psi$

Proposition

$\mathbb{NCLB}$ *is sound with respect to the class of symmetric frames.*

Pseudo-Canonical Model

Definition (Pseudo-Canonical Model)

Define $\mathcal{M}^c = \langle S^c, R^c, V^c \rangle$ as follows:

- $S^c = \{s \mid s \text{ is a maximal consistent set of } \mathbb{NCLB}\}$
- For all $s, t \in S^c$, $sR^c t$ iff there exists χ such that:
  - $\neg\Delta\chi \in s$, and
  - for all φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ implies $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

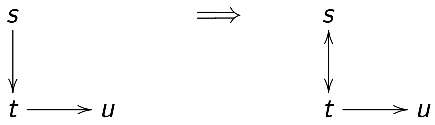
Lemma (Pseudo-Truth Lemma)

For all $\varphi \in$ **NCL** *and* $s \in S^c$, $\mathcal{M}^c, s \vDash \varphi$ *iff* $\varphi \in s$.

R^c is *not* symmetric

Proposition

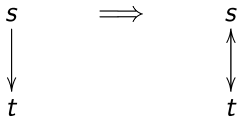
For any $s, t \in S^c$, if $sR^c t$ and $\neg\Delta\chi \in t$ for some χ, then $tR^c s$.



$sR^c t$ iff there exists χ such that:

- $\neg\Delta\chi \in s$, and
- for all φ, $\Delta\varphi \wedge \Delta(\chi \to \varphi) \in s$ implies $\varphi \in t$.

Turn $\mathcal{M}^c$ into a symmetric model



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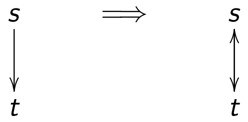
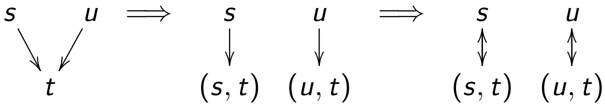


Split the world t:



Canonical Model of $\mathbb{NCLB}$

Definition

The canonical model $\mathcal{M}^+$ of $\mathbb{NCLB}$ is a tuple $\langle S^+, R^+, f, V^+ \rangle$ where:

- $S^+ = \bar{D} \cup \{(s,t) \mid t \in D, sR^c t\}$
- sR^+t iff one of the following cases holds:
  1. $s, t \in \bar{D}$ and $sR^c t$,
  2. $s \in \bar{D}$ and $t = (s, s') \in S^+$,
  3. $t \in \bar{D}$ and $s = (t, t') \in S^+$.
- f is a function assigning each state in S^+ to a maximal consistent set in S^c such that $f(s) = s$ for $s \in \bar{D}$, and $f((s,t)) = t$ for $(s,t) \in S^+$.
- $V^+(p) = \{s \in S^+ \mid p \in f(s)\}$

where $D = \{t \mid t \in S^c, \Delta\chi \in t \text{ for all } \chi, \text{ and there exists an } s \in S^c \text{ such that } sR^c t\}$, where S^c and R^c are defined as in Definition 25, and $\bar{D} = S^c \backslash D$.

f acts like a surjective bounded morphism

Proposition

1. *f is surjective.*
2. *s and $f(s)$ satisfy the same propositional variables.*
3. *if $s \in \bar{D}$ then $sR^{+}t$ implies $f(s)R^{c}f(t)$.*
4. *if $f(s)R^{c}t$ then there exists $u \in S^{+}$ such that $f(u) = t$ and $sR^{+}u$.*

$\mathcal{M}^+$ is desired canonical model

Lemma

$\mathcal{M}^+$ is symmetric.

Proposition

*$\mathcal{M}^+$ preserves the truth values of formulas w.r.t. f. That is: for any $s \in S^+$ and any $\varphi \in$ **NCL**, we have*

$$\mathcal{M}^+, s \vDash \varphi \iff \mathcal{M}^c, f(s) \vDash \varphi.$$

Completeness of $\mathbb{NCLB}$

Theorem

$\mathbb{NCLB}$ *is (sound and) strongly complete with respect to the class of symmetric frames.*

Multimodal **NCL**

- Language ($i \in \mathbf{I}$, where $\mathbf{I}$ is finite)

$$\varphi ::= \top \mid p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Delta_i\varphi \mid \Box_i\varphi$$

$\Box_i\varphi$: φ is necessary for agent i
$\Delta_i\varphi$: φ is non-contingent for agent i, i.e., for i, φ is necessarily true or φ is necessarily false.

- Semantics

$\mathcal{M}, s \vDash \Delta_i\varphi$	$\Leftrightarrow$	for any t_1, t_2 such that sR_it_1, sR_it_2 : $(\mathcal{M}, t_1 \vDash \varphi \Leftrightarrow \mathcal{M}, t_2 \vDash \varphi)$

- Multimodal **NCL** is not normal
- Almost-definability

$$\vDash \nabla_i\psi \to (\Box_i\varphi \leftrightarrow \Delta_i\varphi \wedge \Delta_i(\psi \to \varphi))$$

Proof system for the symmetric frames: Multimodal case

$\mathbb{NCLB}_m$

TAUT	all instances of tautologies
ΔCon	$\Delta_i(\chi \to \varphi) \land \Delta_i(\neg\chi \to \varphi) \to \Delta_i\varphi$
ΔDis	$\Delta_i\varphi \to \Delta_i(\varphi \to \psi) \lor \Delta_i(\neg\varphi \to \chi)$
ΔEqu	$\Delta_i\varphi \leftrightarrow \Delta_i\neg\varphi$
ΔB	$\varphi \to \Delta_i((\Delta_i\varphi \land \Delta_i(\varphi \to \psi) \land \neg\Delta_i\psi) \to \chi)$
MP	From φ and $\varphi \to \psi$ infer ψ
REΔ	From $\varphi \leftrightarrow \psi$ infer $\Delta_i\varphi \leftrightarrow \Delta_i\psi$

Proposition

$\mathbb{NCLB}_m$ *is sound with respect to the class of symmetric frames.*

Pseudo-Canonical Model: again

Definition (Pseudo-Canonical Model)

Define $\mathcal{M}^c = \langle S^c, \{\to_i^c \mid i \in \mathbf{I}\}, V^c\rangle$ as follows:

- $S^c = \{s \mid s$ is a maximal consistent set of $\mathbb{NCLB}_m\}$
- For all $s, t \in S^c$, for all $i \in \mathbf{I}$, $s \to_i^c t$ iff there exists χ such that
  1. $\neg\Delta_i\chi \in s$, and
  2. for all φ, $\Delta_i\varphi \wedge \Delta_i(\chi \to \varphi) \in s$ implies $\varphi \in t$.
- $V^c(p) = \{s \in S^c \mid p \in s\}$.

- Standard multi-modal logic: $s \to_i^c t$ iff $\Box_i\varphi \in s$ implies $\varphi \in t$
- Almost-definability: $\neg\Delta_i\chi \to (\Box_i\varphi \leftrightarrow \Delta_i\varphi \wedge \Delta_i(\chi \to \varphi))$

Pseudo-Truth Lemma: again

Lemma

For all $\varphi \in \mathbf{NCL}$ *and* $s \in S^c$, $\mathcal{M}^c, s \vDash \varphi$ *iff* $\varphi \in s$.

$\rightarrow_i^c$ is *not* symmetric

Proposition

For any $s, t \in S^c$ *and any* $i \in \mathbf{I}$*, if* $s \rightarrow_i^c t$ *and* $t \rightarrow_i^c t'$ *for some* $t' \in S^c$*, then* $t \rightarrow_i^c s$*.*

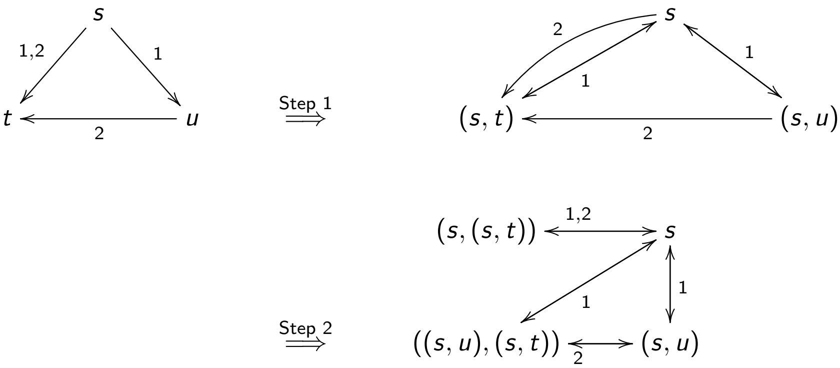
- The canonical model for $\mathbb{NCLB}$ cannot be generalized into $\mathbb{NCLB}_m$
- The dead ends are relative to the agents
- A dead end for agent j may be not a dead end for agent i
- Need new strategy

New Strategy: turn $\mathcal{M}^c$ into a symmetric model

Enumerate all of the agents in **I** as $1, 2, 3, \cdots, m$. Starting from $\mathcal{M}^0 = \mathcal{M}^c$ (we may as well assume that $\mathcal{M}^c$ has run out of Prop. 27), we construct the desired model (call it $\mathcal{M}^m$) in m steps.

- In each step we tackle the *dead ends* for that agent, by replacing those dead ends with some new copies of themselves such that each copy has only one incoming transition for that agent and then adding the back arrows for the agent
- while keeping all the arrows for the other agents in place, with corresponding replacements for the dead ends. We have to provide that
  1. In each step, the accessibility relation for that agent is symmetric,
  2. The symmetry of the previous relation for a fixed agent is not broken, which guarantee $\mathcal{M}^m$ to be symmetric
  3. Each step preserves the truth values of formulas

An example



Canonical model $\mathcal{M}^m$ of $\mathbb{NCLB}_m$

Definition

Define $\mathcal{M}^m = \langle S^m, \{\rightarrow_i^m \mid i \in \mathbf{I}\}, f^m, V^m \rangle$ by induction on $n \leq m$:

- $S^0 = S^c$
- $S^n = \bar{D}_n \cup \{(s, t) \mid t \in D_n \text{ and } s \rightarrow_n^{n-1} t\}$, where $D_n = \{t \mid t \in S^{n-1}, \text{ there is no } t' \in S^{n-1} \text{ such that } t \rightarrow_n^{n-1} t' \text{ and there exists an } s \in S^{n-1} \text{ such that } s \rightarrow_n^{n-1} t\}$, $\bar{D}_n = S^{n-1} \backslash D_n$

Canonical model $\mathcal{M}^m$ of $\mathbb{NCLB}_m$

Definition (Cont')

Define $\mathcal{M}^m = \langle S^m, \{\to_i^m \mid i \in \mathbf{I}\}, f^m, V^m \rangle$ by induction on $n \leq m$:

- $\to_n^0 = \to_n^c$
- $s \to_n^n t$ iff one of the following cases holds:
  1. $s, t \in \bar{D}_n$ and $s \to_n^{n-1} t$,
  2. $s \in \bar{D}_n$ and $t = (s, s') \in S^n$,
  3. $t \in \bar{D}_n$ and $s = (t, t') \in S^n$.
- For $i \neq n$, $s \to_i^n t$ iff one of the following cases holds:
  1. $s, t \in \bar{D}_n$ and $s \to_i^{n-1} t$,
  2. $s \in \bar{D}_n$ and $t = (s'', s') \in S^n$ and $s \to_i^{n-1} s'$,
  3. $t \in \bar{D}_n$ and $s = (t'', t') \in S^n$ and $t' \to_i^{n-1} t$,
  4. $s = (w, v) \in S^n$ and $t = (w', v') \in S^n$ and $v \to_i^{n-1} v'$.

Canonical model $\mathcal{M}^m$ of $\mathbb{NCLB}_m$

Definition (Cont')

Define $\mathcal{M}^m = \langle S^m, \{\rightarrow_i^m \mid i \in \mathbf{I}\}, f^m, V^m \rangle$ by induction on $n \leq m$:

- f^{n+1} is a function from S^{n+1} to S^n such that $f^{n+1}(s) = s$ for $s \in \bar{D}_{n+1}$, and $f^{n+1}((s,t)) = t$ for $(s,t) \in S^{n+1}$
- $V^0(p) = \{s \in S^c \mid p \in s\}$ and $V^{n+1}(p) = \{s \in S^{n+1} \mid f^{n+1}(s) \in V^n(p)\}$

Properties of f^{n+1}

Proposition (Preservation)

Given any $s, t \in S^{n+1}$. If $f^{n+1}(s) \to_i^n f^{n+1}(t)$, then

1. *If $i \neq n+1$, then $s \to_i^{n+1} t$.*
2. *If $i = n+1$, then for some $t' \in S^{n+1}$ such that $s \to_i^{n+1} t'$ and $f^{n+1}(t) = f^{n+1}(t')$.*

Proposition (No Miracle)

Given any $s, t \in S^{n+1}$.

1. *If $i \neq n+1$, then $s \to_i^{n+1} t$ implies $f^{n+1}(s) \to_i^n f^{n+1}(t)$.*
2. *If $i = n+1$ and $s \in \bar{D}_{n+1}$, then $s \to_i^{n+1} t$ implies $f^{n+1}(s) \to_i^n f^{n+1}(t)$.*

For every $n \in [0, m-1]$, f^{n+1} is surjective!

$\mathcal{M}^m$ is symmetric

Proposition

$\mathcal{M}^m$ is symmetric. That is, for all $i \in [1, m]$, $\rightarrow_i^m$ is symmetric:

1. *For every $n \in [1, m]$, $\rightarrow_n^n$ is symmetric.*
2. *If $\rightarrow_i^n$ is symmetric, then $\rightarrow_i^{n+1}$ is also symmetric.*

Truth-preserving in each step

Proposition

For any $n \in [0, m-1]$, *any* $s \in S^{n+1}$, *and any* $\varphi \in \mathbf{NCL}$,

$$\mathcal{M}^{n+1}, s \vDash \varphi \Longleftrightarrow \mathcal{M}^{n}, f^{n+1}(s) \vDash \varphi.$$

Completeness of $\mathbb{NCLB}_m$

Define $f = f^1 \circ f^2 \circ \cdots \circ f^m$.

- $f : S^m \to S^0$ is surjective.
- For any $s \in S^m$ and any $\varphi \in \mathbf{NCL}$, we have

$$\mathcal{M}^m, s \vDash \varphi \iff \varphi \in f(s)$$

Theorem

$\mathbb{NCLB}_m$ is strongly complete with respect to the class of symmetric frames.

Dynamified multimodal **NCL**

- Adding public announcements:

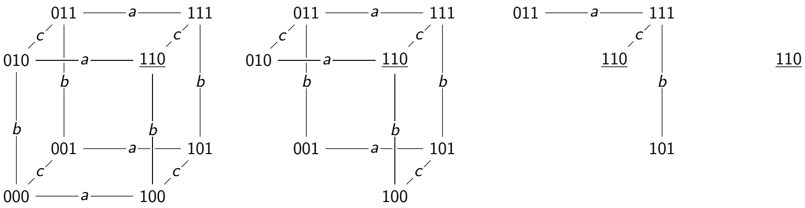
$$[\varphi]\Delta_i\psi \leftrightarrow (\varphi \to (\Delta_i[\varphi]\psi \vee \Delta_i[\varphi]\neg\psi))$$

- Adding action models:

$$[\mathbf{M},\mathbf{s}]\Delta_i\psi \leftrightarrow (\mathbf{pre}(\mathbf{s}) \to \bigwedge_{\mathbf{s}\to_i\mathbf{t}} (\Delta_i[\mathbf{M},\mathbf{t}]\psi \vee \Delta_i[\mathbf{M},\mathbf{t}]\neg\psi))$$

- Completeness, Decidability

Muddy Children Puzzle



$$\mathcal{M}, 110 \vDash [\bigvee_{i=1}^{3} m_i][\bigwedge_{i=1}^{3} \triangledown_i m_i]\neg(\bigwedge_{i=1}^{3} \triangledown_i m_i)$$
$$\mathcal{M}, 110 \vDash [\bigvee_{i=1}^{3} m_i][\bigwedge_{i=1}^{3} \triangledown_i m_i](\bigwedge_{i=1}^{2} \Delta_i m_i \wedge \triangledown_3 m_3)$$
$$\mathcal{M}, 110 \vDash [\bigvee_{i=1}^{3} m_i][\bigwedge_{i=1}^{3} \triangledown_i m_i][\bigwedge_{i=1}^{2} \Delta_i m_i]\Delta_3 m_3$$

Comparison with the known axiomatizations

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6. We extend the results to public announcements and action models, which were not discussed in the literature of non-contingency logic.

Conclusion

1. Almost-definability schema $\nabla\psi \to (\Box\varphi \leftrightarrow \Delta\varphi \wedge \Delta(\psi \to \varphi))$
2. Δ-bisimulation and two characterization fragments
3. Axiomatizations of **NCL** over various frames, via a rather uniform method, where the cases for symmetric frames are highly non-trivial
    - Unimodal
    - Multimodal
4. Dynamic extensions of **NCL**
5. Comparison with the known axiomatizations

Future work

1. Contingency logic and other semantics, such as neighborhood semantics (a manuscript under submission with Hans van Ditmarsch), topological semantics (c.f. [Steinsvold, 2008])
2. The relative succinctness of **NCL** and **ML** on **S5** (in the **K** case, see [van Ditmarsch et al., 2014]).
3. Contingency syllogisms, Contingency logic and First-order logic (c.f. [Brogan, 1967, Béziau, 2000, Read, 2012])
4. Contingency logic and Temporal operators, e.g. future contingents
5. Contingency logic and Epistemology concepts, such as Knowledge, Belief [Costa-Leite, 2007] (general frame classes), Ignorance [van der Hoek and Lomuscio, 2004] and Knowing whether [Fan et al., 2013]
6. Common ignorance, or common knowing whether

Future work (cont')

Arbitrary Knowing Whether Logic (an ongoing joint work with Hans van Ditmarsch)

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Delta\varphi \mid \blacklozenge\varphi$$

- Related to Arbitrary Public Announcement Logic [Balbiani et al., 2007, Balbiani et al., 2008, Balbiani and van Ditmarsch, 2014]
- $\mathcal{M}, s \vDash \blacklozenge\varphi$ iff there is a $\blacklozenge$-free ψ such that $\mathcal{M}, s \vDash \langle !\psi \rangle\varphi$
- Related to Fitch's Knowability Paradox: $\vDash \varphi \to \blacklozenge\Box\varphi$?
- $\nvDash \varphi \to \blacklozenge\Box\varphi$, e.g., Moore-sentence $p \wedge \neg\Box p$
- $\vDash \blacklozenge(\Box\varphi \vee \Box\neg\varphi)$ [van Ditmarsch et al., 2012]
- For every proposition we can get to know whether it is true.
- $\vDash \blacklozenge\Delta\varphi$ (equivalently, $\vDash \neg\blacksquare\nabla\varphi$)
- **AKW** Expressivity? Axiomatization? Decidability?

Future work (cont')

Various kinds of knowledge and their logical representations (beyond 'knowing that'):

- Knowing Whether [Fan et al., 2013]
- 5W1H
  - Knowing What [Wang and Fan, 2013, Wang and Fan, 2014]
  - Knowing Who
  - Knowing When
  - Knowing Where
  - Knowing Why
  - Knowing How

Publications and Submissions

- Publications:
  1. J. Fan, Y. Wang and H. van Ditmarsch. Almost Necessary. Advances in Modal Logic, Volume 10, pages 178-196, 2014.
  2. H. van Ditmarsch, J. Fan, W. van der Hoek, P. Iliev. Some Exponential Lower Bounds on Formula-size in Modal Logic. Advances in Modal Logic, Volume 10, pages 139-157, 2014.
  3. Y. Wang and J. Fan. Conditionally Knowing What. Advances in Modal Logic, Volume 10, pages 569-587, 2014.
  4. Y. Wang and J. Fan. Knowing That, Knowing What, and Public Communication: Public Announcement Logic with Kv Operators, Proc. of 23rd IJCAI, pages 1147-1154, 2013.
  5. J. Fan, Y. Wang and H. van Ditmarsch. Contingency and Knowing Whether. To appear in *The Review of Symbolic Logic*.
  6. Y. Wang and J. Fan. Epistemic Informativeness. To appear in the Proceedings of the Second Asian Workshop on Philosophical Logic.

Publications and Submissions (cont')

- Submissions:
  1. J. Fan and H. van Ditmarsch. Neighborhood Contingency Logic. Submitted to Indian Conference on Logic and its Applications (ICLA 2015), August 2014.

CELLO, LORIA-CNRS/Université de Lorraine

Computational Epistemic Logic in LOrraine, led by Hans van Ditmarsch, who is the holder of ERC starting grant 313360 EPS, called Epistemic Protocol Synthesis.

Activities

- personal meeting between Hans van Ditmarsch and I, for my research progress (once a week)
- CELLO group meeting, for Hans's project (once a week)
- Academic conferences
  - 10th Advances in Modal Logic (AiML 2014), August 5 – August 8, University of Groningen, The Netherlands. Presentation "Almost Necessary".
  - 26th European Summer School in Logic, Language and Information (ESSLLI 2014), August 11 – August 22, University of Tübingen, Germany.

PKU & RuG: AiML 2014

Thank you for your attention!

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